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A Brief Introduction to Quantum Energy Teleportation

Masahiro Hotta

Department of Physics, Faculty of Science, Tohoku University, Sendai 980-8578, Japan
hotta@tuhep.phys.tohoku.ac.jp

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Abstract

A new quantum protocol is introduced which attains energy transportation only by local operations and classical communication retaining all physical laws including local energy conservation.

1 Introduction

Energy transportation is a basic process for not only applied technology but also fundamental physics. Usual methods of the transportation require physical carriers of energy like electric currents and photons. For example, let us consider a typical energy transportation channel like an electric cable and an optical fiber. For the transportation, we must first infuse energy to a gate edge A of the channel and excite energy carriers. Eventually, the energy carriers move to an outlet edge B of the channel by time evolution of the channel dynamics. After the carriers arrive at B, we can extract energy from the carriers and harness it for any purpose. If the channel is in the ground state, no activated energy carriers exist around B. Hence, by using the usual methods, we cannot extract energy from B in the ground state.

This ground-state aspect of the usual transportation essentially remains unchanged even if quantum effect is taken account of. In quantum theory, we have nonvanishing zero-point energy of quantum fluctuation even in the ground state. However, as well known, this zero-point energy at B cannot be extracted by local operations at B. Inversely the local operations excite the quantum fluctuation by infusing energy into the channel.

Amazingly, the situation drastically changes by adopting local operations and classical communication of new quantum protocols called quantum energy teleportation (QET for short) [1]-[8]. If we locally measure quantum fluctuation at A in the ground state and announce the measurement result to B at a speed much faster than the velocity of energy carriers, a part of the zero-point energy at B can be extracted by a local operation dependent on the measurement result before the arrival of energy carriers.

This QET protocol retains all physical laws including local energy conservation. By emitting positive energy $+E_B$ to outside systems, the zero-point fluctuation at B of the channel can be more suppressed than that of the ground state, yielding negative energy $-E_B$ at B. Here we fix the origin of the energy density of the channel such that the expectational value vanishes for the ground state. Thus the total energy of the channel is nonnegative. In general, quantum interference among total energy eigenstates can produce various states containing regions of such negative energy density, although the total energy remains nonnegative.

The above local measurement at A changes the quantum state. The post-measurement state of the channel is not the ground state but instead

an excited state with positive energy E_A. Therefore the same amount of energy E_A must be infused into A by the measurement device, respecting local energy conservation law. This energy is regarded as energy input of the teleportation. Meanwhile, the extracted energy E_B from B is regarded as energy output of the teleportation.

The root of the protocol is a correlation between the measurement information at A and the quantum fluctuation at B via the ground-state entanglement. Due to the correlation, we are able to estimate the quantum fluctuation at B based on the announced measurement result and devise a strategy to suppress the quantum fluctuation at B. During the selected operation on quantum fluctuation at B generating negative energy $-E_B$, surplus positive energy $+E_B$ is transferred from B to external systems layed at the region of B. Therefore, QET increases not the total energy at the region of B but instead the percentage of available energy at the region of B to be harnessed for arbitrary purposes by decreasing the zero-point energy of B.

Physical energy carriers do not play any role for the energy extraction during this short-time QET process. Soon after a one-round completion of the protocol, the input energy E_A still exists at A because late-time evolution of the energy carries does not begin yet. Let us imagine that we attempt to completely withdraw E_A by local operations at A after the extraction of energy from B. If this was possible, the energy gain E_B might have no cost. However, if so, the total energy of the channel became equal to $-E_B$ and negative. Meanwhile, we know that the total energy of the system is nonnegative by our definition of the origin of the energy density. Hence, it is not allowed physically to withdraw energy larger than $E_A - E_B$ by local operations at A. This argument also implies that E_A is lower bounded by E_B. Another reason for this inability of complete extraction of E_A is because the first measurement made at A breaks the ground-state entanglement between quantum fluctuation at A and quantum fluctuation at B. Therefore, after the measurement at A, the ground state (zero-energy state) is no longer recovered only by A's local operations, which do not restore the above broken entanglement. Hence it can be concluded that a part of input energy E_A cannot be extracted from A during the short time scale. QET enables this residual energy at A to be effectively extracted in part as E_B from the distant point B by use of the measurement information of A. It seems like, treating the input energy E_A as a "pawn", the quantum system "pays" the output energy $+E_B$ by doing bookkeeping with a record of negative value of energy, $-E_B$. Needless to say, we can harness the extracted energy $+E_B$ freely and

do not need to return it to the quantum system. After the completion of the QET process, a part of the positive energy E_A at A compensates for the negative energy $-E_B$ at B during the late-time evolution of the energy-carrier dynamics.

Another way of saying QET is possible. The zero-point energy at B in the ground state is not accessible by local operations. This looks like the energy is saved in a locked safe under ground. In QET, we get information about the key to open the safe by a remote measurement at A via entanglement. However, we must then pay for it to A. The cost is energy E_A, which is larger than the zero-point energy E_B extracted from the safe at B.

It is worth noting that, in QET, energy can be also extracted simultaneously from other subsystems C, D, $\cdots$ if we know the measurement result of A. Therefore, more strictly speaking, E_A is lower bounded by sum of all of the possible energy extraction, $E_B + E_C + E_D + \cdots$. In effect, the input energy E_A is stored in the quantum system with a form like broadened oil field [1].

The QET protocols can be implemented, at least theoretically, to various physical systems, including spin chains [1]-[2], cold trapped ions [3], quantum fields [4]-[6] and linear harmonic chains [7]. Recently, a nontrivial QET protocol has been proposed for a minimal model [8]. In this presentation, analysis of the minimal QET protocol is given.

2 Minimal QET Model

The minimal model [8] is defined as follows. The system consists of two qubits A and B. Its Hamiltonian reads $H = H_A + H_B + V$, where each contribution is given by

$$H_A = h\sigma_A^z + \frac{h^2}{\sqrt{h^2+k^2}}, \tag{1}$$

$$H_B = h\sigma_B^z + \frac{h^2}{\sqrt{h^2+k^2}}, \tag{2}$$

$$V = 2k\sigma_A^x\sigma_B^x + \frac{2k^2}{\sqrt{h^2+k^2}}, \tag{3}$$

and h and k are positive constants with energy dimensions, σ_A^x (σ_B^x) is the x-component of the Pauli operators for the qubit A (B), and σ_A^z (σ_B^z) is

the z-component for the qubit A (B). The constant terms in Eqs. (1)-(3) are added in order to make the expectational value of each operator zero for the ground state $|g\rangle$: $\langle g|H_A|g\rangle = \langle g|H_B|g\rangle = \langle g|V|g\rangle = 0$. Because the lowest eigenvalue of the total Hamiltonian H is zero, H is a nonnegative operator: $H \geq 0$. Meanwhile, it should be noticed that H_B and $H_B + V$ have negative eigenvalues, which can yield negative energy density at B. The ground state is given by

$$
\begin{aligned}
|g\rangle = & \frac{1}{\sqrt{2}}\sqrt{1-\frac{h}{\sqrt{h^2+k^2}}}|+\rangle_A|+\rangle_B \\
& -\frac{1}{\sqrt{2}}\sqrt{1+\frac{h}{\sqrt{h^2+k^2}}}|-\rangle_A|-\rangle_B,
\end{aligned}
$$

where $|\pm\rangle_A$ $(|\pm\rangle_B)$ is the eigenstate of σ_A^z (σ_B^z) with eigenvalue ± 1. A QET protocol is constructed by the following three steps:

- I. A projective measurement of observable σ_A^x is performed to A in the ground state $|g\rangle$ and a measurement result $(-1)^\mu$ with $\mu = 0, 1$ is obtained. During the measurement, positive amount of energy

$$
E_A = \frac{h^2}{\sqrt{h^2+k^2}} \tag{4}
$$

  is infused to A on average.

- II. The result μ is announced to B via a classical channel at a speed much faster than the velocity of energy diffusion of the system.

- III. Let us consider a local unitary operation of B depending on the value of μ given by $U_B(\mu) = I_B \cos\theta + i\,(-1)^\mu\,\sigma_B^y \sin\theta$, where θ is a real constant which satisfies

$$
\begin{aligned}
\cos(2\theta) &= \frac{h^2+2k^2}{\sqrt{\left(h^2+2k^2\right)^2+h^2k^2}}, \\
\sin(2\theta) &= -\frac{hk}{\sqrt{\left(h^2+2k^2\right)^2+h^2k^2}}.
\end{aligned}
$$

$U_B(\mu)$ is performed on B. During the operation, positive amount of energy

$$E_B = \frac{h^2 + 2k^2}{\sqrt{h^2 + k^2}} \left[\sqrt{1 + \frac{h^2 k^2}{(h^2 + 2k^2)^2}} - 1 \right] \tag{5}$$

is extracted from B on average.

The outline of derivation of E_A and E_B is given in Appendix I. The nontrivial feature of this model is that the measurement performed at A does not increase the average energy of B at all. By explicit calculations using $[\sigma_A^x, H_B] = [\sigma_A^x, V] = 0$, the average values of H_B and V are found to remain zero after the measurement and are the same as those of the ground state. Thus, we cannot extract energy from B only by local operations soon after the measurement of A. Even though energy carriers coming from A have not arrived at B yet, the QET protocol is able to achieve energy extraction from B. As mentioned above, this success of energy extraction is achieved by emergence of negative energy density at B. Finally, it is noted that decrease of ground-state entanglement between A and B by the measurement at A has a natural connection with the amount of energy teleported from A to B (Appendix II).

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A Appendix I. Outline of Derivation of E_A and E_B

In this appendix, outline of derivation of Eq. (4) and Eq. (5) is given [8]. Besides, nontrivial point of the minimal QET model is stressed.

Firstly, the projection operator corresponding to each measurement result $(-1)^\mu$ of σ_A^x is given by

$$P_A(\mu) = \frac{1}{2}\left(1 + (-1)^\mu \sigma_A^x\right).$$

The post-measurement state of the two qubits with output μ is given by

$$|A(\mu)\rangle = \frac{1}{\sqrt{\langle g|P_A(\mu)|g\rangle}} P_A(\mu)|g\rangle,$$

where $\langle g|P_A(\mu)|g\rangle$ is appearance probability of μ for the ground state. It is easy to check that the average post-measurement state given by

$$\sum_\mu \langle g|P_A(\mu)|g\rangle |A(\mu)\rangle\langle A(\mu)| = \sum_\mu P_A(\mu)|g\rangle\langle g|P_A(\mu)$$

has a positive expectational value E_A of H, which energy distribution is localized at A. In fact, the value defined by

$$E_A = \sum_\mu \langle g|P_A(\mu) H P_A(\mu)|g\rangle$$

is computed straightforwardly as

$$E_A = \sum_\mu \langle g|P_A(\mu) H_A P_A(\mu)|g\rangle = \frac{h^2}{\sqrt{h^2+k^2}}. \tag{6}$$

Thus Eq.(4) is obtained. This infused energy E_A is regarded as the QET energy input via the measurement of A. During the measurement, E_A is transferred from external systems including the measurement device with a battery respecting local energy conservation. The QET energy conservation law during local measurements has been discussed in [2].

Because energy of B remains zero after the measurement, we cannot extract energy from B by local operations soon after the measurement. To verify this fact explicitly, let us consider any local unitary operation W_B which is **independent** of A's measurement result and performed on B. Then, the post-operation state ω is given by

$$\omega = \sum_\mu W_B P_A(\mu)|g\rangle\langle g|P_A(\mu) W_B^\dagger.$$

The energy difference after the operation is calculated as

$$E_A - \mathrm{Tr}\left[\omega H\right] = -\langle g|W_B^\dagger \left(H_B + V\right) W_B|g\rangle, \tag{7}$$

where we have used

$$W_B^\dagger H_A W_B = H_A W_B^\dagger W_B = H_A,$$

$$\left[W_B^\dagger \left(H_B + V\right) W_B,\ P_A(\mu)\right] = 0,$$

and the completeness relation of $P_A(\mu)$:

$$\sum_\mu P_A(\mu) = 1_A.$$

From Eq. (7), it is proven that the energy difference is nonpositive:

$$E_A - \mathrm{Tr}\left[\omega H\right] = -\langle g|W_B^\dagger H W_B|g\rangle \leq 0,$$

because of a relation such that $\langle g|W_B^\dagger H_A W_B|g\rangle = \langle g|H_A|g\rangle = 0$ and the nonnegativity of H. Therefore, as a natural result, no local operation on B

independent of μ extracts positive energy from B by decreasing total energy of the two qubits.

After a while, the infused energy E_A diffuses to B. The time evolution of the expectational values H_B and V of the average post-measurement state is calculated as

$$\begin{aligned}\langle H_B(t)\rangle &= \sum_\mu \langle g|P_A(\mu)|g\rangle\langle A(\mu)|e^{itH}H_B e^{-itH}|A(\mu)\rangle \\ &= \frac{h^2}{2\sqrt{h^2+k^2}}\left[1-\cos\left(4kt\right)\right],\end{aligned}$$

and $\langle V(t)\rangle = 0$. Therefore, energy can be extracted from B after a diffusion time scale of $1/k$; this is just a usual energy transportation from A to B. The QET protocol can transport energy from A to B in a time scale much shorter than that of this usual transportation. In the protocol, the measurement output μ is announced to B. Because the model is non-relativistic, the propagation speed of the announced output can be much faster than the diffusion speed of the infused energy and can be approximated as infinity. Soon after the arrival of the output μ, we perform $U_B(\mu)$ on B dependent on μ. Then, the average state after the operation is given by

$$\rho = \sum_\mu U_B(\mu)P_A(\mu)|g\rangle\langle g|P_A(\mu)U_B(\mu)^\dagger.$$

The expectational value of the total energy after the operation is given by

$$\mathrm{Tr}\left[\rho H\right] = \sum_\mu \langle g|P_A(\mu)U_B(\mu)^\dagger H U_B(\mu)P_A(\mu)|g\rangle.$$

On the basis of the fact that $U_B(\mu)$ commutes with H_A and Eq. (6), E_B is computed as

$$E_B = E_A - \mathrm{Tr}\left[\rho H\right] = -\mathrm{Tr}\left[\rho\left(H_B+V\right)\right].$$

Further, on the basis of the fact that $P_A(\mu)$ commutes with $U_B(\mu)$, H_B and V, the energy can be written as

$$E_B = -\sum_\mu \langle g|P_A(\mu)\left(H_B(\mu)+V(\mu)\right)|g\rangle,$$

where the μ-dependent operators are given by $H_B(\mu) = U_B(\mu)^\dagger H_B U_B(\mu)$ and $V(\mu) = U_B(\mu)^\dagger V U_B(\mu)$. By straightforward calculation, E_B is computed as

$$E_B = -\frac{1}{\sqrt{h^2+k^2}}\left[\left(h^2+2k^2\right)\left[1-\cos\left(2\theta\right)\right]+hk\sin(2\theta)\right]. \tag{8}$$

Note that $E_B = 0$ if $\theta = 0$, as it should be. If we take a small negative value of θ in Eq. (8), it is noticed that E_B takes a small positive value such that

$$E_B \sim \frac{2hk\,|\theta|}{\sqrt{h^2+k^2}} > 0.$$

Maximization of E_B in terms of θ is attained by taking a value of θ which satisfies

$$\cos(2\theta) = \frac{h^2+2k^2}{\sqrt{\left(h^2+2k^2\right)^2 + h^2k^2}},$$
$$\sin(2\theta) = -\frac{hk}{\sqrt{\left(h^2+2k^2\right)^2 + h^2k^2}}.$$

Substituting these relations into Eq. (8) yields the positive value of E_B in Eq. (5).

The measurement of A can be extended to POVM measurements. Let S_{M_A} denote a set of POVM measurements for A which measurement operators $M_A(\mu)$ with measurement output μ commute with the interaction Hamiltonian V. The measurement operator $M_A(\mu)$ takes the form of

$$M_A(\mu) = e^{i\delta_\mu}\left(m_\mu + e^{i\alpha_\mu} l_\mu \sigma_A^x\right). \tag{9}$$

The coefficients m_μ, l_μ, α_μ and δ_μ are real constants which satisfy

$$\sum_\mu \left(m_\mu^2 + l_\mu^2\right) = 1,$$
$$\sum_\mu m_\mu l_\mu \cos\alpha_\mu = 0.$$

The POVM corresponding to $M_A(\mu)$ is defined by

$$\Pi_A(\mu) = M_A(\mu)^\dagger M_A(\mu),$$

which satisfies the completeness relation,

$$\sum_\mu \Pi_A(\mu) = 1_A.$$

By introducing the emergence probability $p_A(\mu) = \langle g|\Pi_A(\mu)|g\rangle$ of output μ for the ground state and a real parameter $q_A(\mu)$, the POVM is written as follows:

$$\Pi_A(\mu) = p_A(\mu) + q_A(\mu)\sigma_A^x.$$

By taking suitable values of m_μ, l_μ, and α_μ, all values of $p_A(\mu)$ and $q_A(\mu)$ are permissible as long as they satisfy $\sum_\mu p_A(\mu) = 1$, $\sum_\mu q_A(\mu) = 0$ and $p_A(\mu) \geq |q_A(\mu)|$. The post-measurement state of the two qubits with output μ is given by

$$|A'(\mu)\rangle = \frac{1}{\sqrt{p_A(\mu)}} M_A(\mu)|g\rangle. \tag{10}$$

This measurement excites the system. Input energy E_A of QET in this case is defined by

$$E_A = \sum_\mu \langle g|M_A(\mu)^\dagger H M_A(\mu)|g\rangle$$

and is computed as

$$E_A = \frac{2h^2}{\sqrt{h^2+k^2}} \sum_\mu l_\mu^2.$$

It is also possible to generalize the operation of B as

$$U'_B(\mu) = I_B \cos\omega_\mu + i\vec{n}_\mu \cdot \vec{\sigma}_B \sin\omega_\mu, \tag{11}$$

where ω_μ is a real parameter, $\vec{n}_\mu = (n_{x\mu}, n_{y\mu}, n_{z\mu})$ is a three-dimensional unit real vector and $\vec{\sigma}_B$ is the Pauli spin vector operator of B. After the operation of B, the average state becomes

$$\rho' = \sum_\mu U'_B(\mu) M_A(\mu)|g\rangle\langle g|M_A(\mu)^\dagger U'_B(\mu)^\dagger.$$

Output energy E_B of QET is defined by

$$E_B = E_A - \mathrm{Tr}\left[\rho' H\right]$$

and computed as

$$E_B = \frac{1}{\sqrt{h^2+k^2}} \sum_\mu Q(\mu), \tag{12}$$

where $Q(\mu)$ is given by

$$Q(\mu) = X(\mu)\cos(2\omega_\mu) - hkq_A(\mu)n_{y\mu}\sin(2\omega_\mu) - X(\mu),$$

and $X(\mu)$ is defined by

$$X(\mu) = p_A(\mu) \left[h^2 \left(1 - n_{z\mu}^2\right) + 2k^2 \left(1 - n_{x\mu}^2\right)\right] - 3hkq_A(\mu)n_{x\mu}n_{z\mu}.$$

It can proven that, for each measurement belonging to S_{M_A}, an operation $U'_B(\mu)$ properly dependent on $M_A(\mu)$ and μ always yields a positive value of E_B [8].

B Appendix II. Energy-Entanglement Relation for Minimal QET Model

In this appendix, we analyze entanglement breaking by the measurement of A and show two inequalities between entanglement consumption in the measurement and amount of teleported energy [8]. We adopt entropy of entanglement as a quantitative measure of entanglement. The entropy of a pure state $|\Psi_{AB}\rangle$ of A and B is defined as

$$S_{AB} = -\operatorname*{Tr}_B \left[\operatorname*{Tr}_A \left[|\Psi_{AB}\rangle\langle\Psi_{AB}|\right] \ln \operatorname*{Tr}_A \left[|\Psi_{AB}\rangle\langle\Psi_{AB}|\right]\right].$$

Before the measurement, the total system is prepared to be in the ground state $|g\rangle$. The reduced state of B is given by

$$\rho_B = \operatorname*{Tr}_A \left[|g\rangle\langle g|\right].$$

After the POVM measurement outputting μ defined by Eq. (9), the state is transferred into a pure state $|A'(\mu)\rangle$ in Eq. (10). The reduced post-measurement state of B is calculated as

$$\rho_B(\mu) = \frac{1}{p_A(\mu)} \operatorname*{Tr}_A \left[\Pi_A(\mu)|g\rangle\langle g|\right].$$

The entropy of entanglement of the ground state is given by

$$S_{AB}(g) = -\operatorname*{Tr}_B \left[\rho_B \ln \rho_B\right]$$

and that of the post-measurement state with output μ is given by

$$S_{AB}(\mu) = -\operatorname*{Tr}_B \left[\rho_B(\mu) \ln \rho_B(\mu)\right].$$

By using these results, we define the consumption of ground-state entanglement by the measurement as the difference between the ground-state entanglement and the averaged post-measurement-state entanglement:

$$\Delta S_{AB} = S_{AB}(g) - \sum_{\mu} p_A(\mu) S_{AB}(\mu).$$

Interestingly, this quantity is tied to the quantum mutual information between the measurement result of A and the post-measurement state of B. Let us introduce a Hilbert space for a measurement pointer system $\bar{A}$ of the POVM measurement, which is spanned by orthonormal states $|\mu_{\bar{A}}\rangle$ corresponding to the output μ satisfying $\langle \mu_{\bar{A}} | \mu'_{\bar{A}} \rangle = \delta_{\mu\mu'}$. Then, the average state of $\bar{A}$ and B after the measurement is given by

$$\Phi_{\bar{A}B} = \sum_{\mu} p_A(\mu) |\mu_{\bar{A}}\rangle\langle \mu_{\bar{A}}| \otimes \rho_B(\mu).$$

By using the reduced operators $\Phi_{\bar{A}} = \mathrm{Tr}_B \left[\Phi_{\bar{A}B}\right]$ and $\Phi_B = \mathrm{Tr}_{\bar{A}} \left[\Phi_{\bar{A}B}\right]$, the mutual information $I_{\bar{A}B}$ is defined as

$$I_{\bar{A}B} = -\operatorname*{Tr}_{\bar{A}} \left[\Phi_{\bar{A}} \ln \Phi_{\bar{A}}\right] - \operatorname*{Tr}_{B} \left[\Phi_B \ln \Phi_B\right] + \operatorname*{Tr}_{\bar{A}B} \left[\Phi_{\bar{A}B} \ln \Phi_{\bar{A}B}\right].$$

By using $\mathrm{Tr}_B \left[\Phi_{\bar{A}B}\right] = \sum_\mu p_A(\mu) |\mu_{\bar{A}}\rangle\langle \mu_{\bar{A}}|$ and $\mathrm{Tr}_{\bar{A}} \left[\Phi_{\bar{A}B}\right] = \sum_\mu p_A(\mu)\rho_B(\mu) = \rho_B$, it can be straightforwardly proven that $I_{\bar{A}B}$ is equal to ΔS_{AB}. This relation provides another physical interpretation of ΔS_{AB}.

Next, let us calculate ΔS_{AB} explicitly. All the eigenvalues of $\rho_B(\mu)$ are given by

$$\lambda_{\pm}(\mu) = \frac{1}{2}\left[1 \pm \sqrt{\cos^2\varsigma + \sin^2\varsigma \frac{q_A(\mu)^2}{p_A(\mu)^2}}\right], \tag{13}$$

where ς is a real constant which satisfies

$$\cos\varsigma = \frac{h}{\sqrt{h^2+k^2}}, \ \ \sin\varsigma = \frac{k}{\sqrt{h^2+k^2}}.$$

The eigenvalues of ρ_B are obtained by substituting $q_A(\mu) = 0$ into Eq. (13). By using $\lambda_s(\mu)$, ΔS_{AB} can be evaluated as

$$\Delta S_{AB} = \sum_{\mu} p_A(\mu) f_I \left(\frac{q_A(\mu)^2}{p_A(\mu)^2}\right), \tag{14}$$

where $f_I(x)$ is a monotonically increasing function of $x \in [0,1]$ and is defined by

$$
\begin{aligned}
f_I(x) = & \frac{1}{2}\left(1+\sqrt{\cos^2\varsigma + x\sin^2\varsigma}\right) \\
& \times \ln\left(\frac{1}{2}\left(1+\sqrt{\cos^2\varsigma + x\sin^2\varsigma}\right)\right) \\
& + \frac{1}{2}\left(1-\sqrt{\cos^2\varsigma + x\sin^2\varsigma}\right) \\
& \times \ln\left(\frac{1}{2}\left(1-\sqrt{\cos^2\varsigma + x\sin^2\varsigma}\right)\right) \\
& - \frac{1}{2}(1+\cos\varsigma)\ln\left(\frac{1}{2}(1+\cos\varsigma)\right) \\
& - \frac{1}{2}(1-\cos\varsigma)\ln\left(\frac{1}{2}(1-\cos\varsigma)\right).
\end{aligned}
$$

It is worth noting [8] that the maximum of E_B of Eq. (12) in terms of $U'_B(\mu)$ of Eq. (11) takes a form similar to Eq. (14) as

$$
\max_{U'_B(\mu)} E_B = \sum_\mu p_A(\mu) f_E\left(\frac{q_A(\mu)^2}{p_A(\mu)^2}\right), \tag{15}
$$

where $f_E(x)$ is a monotonically increasing function of $x \in [0,1]$ and is defined by

$$
f_E(x) = \sqrt{h^2+k^2}\left(1+\sin^2\varsigma\right)\left[\sqrt{1+\frac{\cos^2\varsigma\sin^2\varsigma}{\left(1+\sin^2\varsigma\right)^2}x} - 1\right].
$$

Expanding both $f_I(x)$ and $f_E(x)$ around $x = 0$ yields

$$
\begin{aligned}
f_I(x) &= \frac{\sin^2\varsigma}{4\cos\varsigma}\ln\frac{1+\cos\varsigma}{1-\cos\varsigma}x + O(x^2), \\
f_E(x) &= \sqrt{h^2+k^2}\frac{\cos^2\varsigma\sin^2\varsigma}{2\left(1+\sin^2\varsigma\right)}x + O(x^2).
\end{aligned}
$$

By deleting x in the above two equations, we obtain the following relation for weak measurements with infinitesimally small $q_A(\mu)$:

$$
\Delta S_{AB} = \frac{1+\sin^2\varsigma}{2\cos^3\varsigma}\ln\frac{1+\cos\varsigma}{1-\cos\varsigma}\frac{\max_{U'_B(\mu)} E_B}{\sqrt{h^2+k^2}} + O(q_A(\mu)^4).
$$

It is of great significance [8] that this relation can be extended as the following inequality for general measurements of S_{M_A} :

$$\Delta S_{AB} \geq \frac{1+\sin^2\varsigma}{2\cos^3\varsigma}\ln\frac{1+\cos\varsigma}{1-\cos\varsigma}\frac{\max_{U'_B(\mu)} E_B}{\sqrt{h^2+k^2}}. \tag{16}$$

This inequality implies that a large amount of teleported energy requests a large amount of consumption of the ground-state entanglement between A and B. In addition, we can prove another inequality between the teleported energy and the entanglement consumption [8]. The following inequality is satisfied for all measurements of S_{M_A}:

$$\begin{aligned} &\max_{U'_B(\mu)} E_B \\ &\geq \frac{2\sqrt{h^2+k^2}\left[\sqrt{4-3\cos^2\varsigma}-2+\cos^2\varsigma\right]}{(1+\cos\varsigma)\ln\left(\frac{2}{1+\cos\varsigma}\right)+(1-\cos\varsigma)\ln\left(\frac{2}{1-\cos\varsigma}\right)} \\ &\times \Delta S_{AB}. \end{aligned} \tag{17}$$

This ensures that if we have consumption of ground-state entanglement ΔS_{AB} for a measurement of S_{M_A}, we can in principle teleport energy from A to B, where the energy amount is greater than the value of the right-hand-side term of Eq. (17). This bound is achieved for non-zero energy transfer by measurements with $q_A(\mu) = \pm p_A(\mu)$. The inequalities in Eq. (16) and Eq. (17) help us to gain a deeper understanding of entanglement as a physical resource because they show that the entanglement decrease by the measurement of A is directly related to the increase of the available energy at B as an evident physical resource.